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Equal Functions

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Introduction

In this unit, we will build on our understanding of order of operations to include operations with variables. Using this new understanding, we will learn to recognize and build *equal functions*.

Consider the following two functions:

```
to f :x
op 2 * (:x + 3)
end

to g :x
op 2 * :x + 6
end
```

Here's what happens when we execute the operations with different inputs:

```
show f 0
6
show g 0
6
show f 1
8
show g 1
8
show f 2
10
show g 2
10
show f 10
26
show g 10
26
show f 100
206
show g 100
206
```

Notice that when f and g are given the same input, they return the same output. Will they always give the same output whenever they are given the same input? This is a question we will investigate in this unit.

In general, two functions f and g are equal if $f(x) = g(x)$ for any value of x.

Number Properties

Number properties are special qualities that numbers have that can help us recognize when two functions are equal. The properties we discuss in this unit are the *commutative property*, the *associative property*, and the *distributive property*.

Commutative Property

The commutative property of addition says that we can add numbers together in any order, so:

$3 + 5 = 5 + 3$
$y + 9 = 9 + y$
$p + 3q = 3q + p$

In general, adding x and y means something like "put together x things with y things and then count them all up and see how many you have", or "walk x units and then walk y more units in the same direction and see how far you walked all together", so it doesn't matter if we put x first or y first.

In Logo, the commutative property of addition means that the following operations are equal:

```
to f :x
op :x + 3
end

to g :x
op 3 + :x
end
```

In other words, if $f(x) = x + 3$ and $g(x) = 3 + x$, then $f(x) = g(x)$.

Subtraction is *not* commutative. The idea of subtraction could be stated as "put together a collection of x things and then give away y of them and see how many are left", or "walk x units to the left and then turn around and walk y units to the right and see how far left or right you are from your starting place". For subtraction, order matters, so:

$4 - 2 \neq 2 - 4$
$10 - z \neq z - 10$ (unless $z = 10$)
$5m - n \neq n - 5m$ (unless $5m = n$)

The "$\neq$" sign is read as "does not equal". Expressions with this sign are called *inequalities*.

Commutative Property (continued)

For the inequality $10 - z \neq z - 10$, why did I write “unless $z = 10$”?

Consider

```
to f :z
op 10 - :z
end

to g :z
op :z - 10
end
```

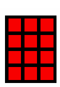
If we execute these procedures with inputs of zero, we get the same inputs for both:

```
show f 10
0
show g 10
0
```

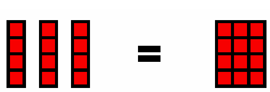
But if we use any other number as input for both, the two functions give different answers (we’ll take a look at subtractions like 3 – 10 in later units).

Multiplication is also commutative. Let’s see why.

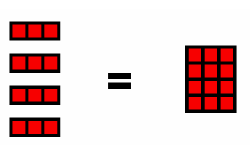
At some point in elementary school, you probably saw arrays used to represent multiplication. For example, here is an array of twelve squares:



We can think of this array as representing $3 \times 4 = 12$



or $4 \times 3 = 12$



⋮

Commutative Property (continued)

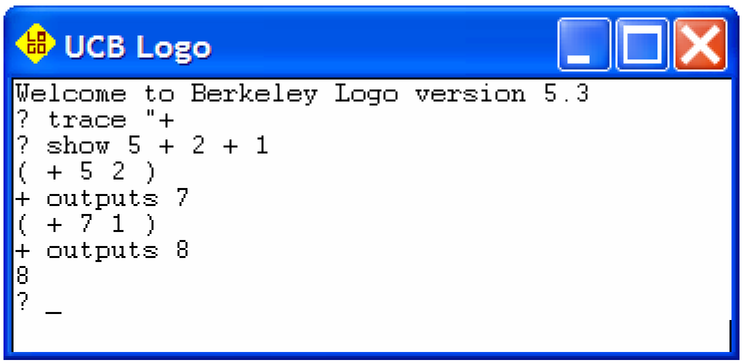
When arrays are made of squares that form rectangles with no spaces in between the squares, we can also consider the squares as units of measurement for the area of the rectangle. We can say that this kind of 4×3 array is a rectangle with an *area* of 12 units.

Division is *not* commutative. For example,

$12 \div 3 \neq 3 \div 12$
$p \div 7 \neq 7 \div p$ (unless $p = 7$)

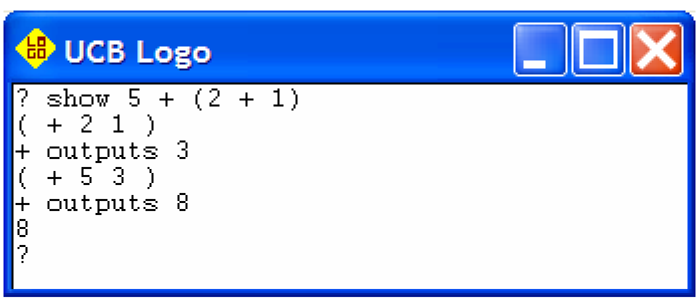
Associative Property

In the last unit, we saw that when we have more than one addition in an instruction, the additions are evaluated from left to right. To give you an example, I put a trace on the + primitive and executed an instruction with two additions:



One cool thing about Berkeley Logo is that it allows you to trace primitives, which you can't do in MSWLogo. Anyway, you can see from the trace above that we first added the 5 and the 2 to get 7 and then added the 7 to the 1.

We can change the order of evaluation with parentheses:



The associative property of addition basically states that you can add from left to right or from right to left.

$(5 + 2) + 1 = 5 + (2 + 1)$
$(a + b) + c = a + (b + c)$

What is the difference between this property and the commutative property? In the commutative property, you can actually change the order of the numbers or variables added. In the associative property, you change the order of evaluation.

Subtraction is *not* associative, multiplication *is* associative, division is *not* associative.

Associative Property (continued)

By the associative property of multiplication, these two Logo operations are equal:

```
to f :x
op (2 * 4) * :x
end

to g :x
op 2 * (4 * :x)
end
```

Here's how this equality would be expressed in standard algebra:

$f(x) = (2 \times 4)x$
$g(x) = 2(4x)$

By the associative property, we know that

$(2 \times 4)x = 2(4x)$

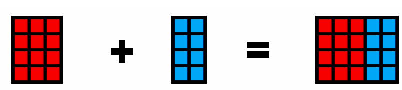
so $f(x) = g(x)$

Try out both Logo functions. You will see that when they both get the same input, they create the same output.

Distributive Property

In the section on the commutative property, we reviewed the use of arrays to represent multiplication and saw how arrays of squares can be used to measure the area of a rectangle.

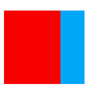
We can also add arrays together. Adding arrays reveals another important property of numbers that can help us recognize equal functions. Let's add a 4×3 array to a 4×2 array:



We can see that the resulting array is 4×5 which shows us that:

$$4 \times 3 + 4 \times 2 = 4 \times (3 + 2)$$

Now let's look at the general case:



If the red rectangle above is $a \times b$ (a units high and b units wide) and the blue rectangle is $a \times c$ (a units high and c units wide), then we see that

$$ab + ac = a(b + c)$$

or

$$a(b + c) = ab + ac$$

Let's use the distributive property to compare two Logo functions:

```
to f :x
op 2 * (:x + 3)
end

to g :x
op 2 * :x + 6
end
```

⋮

Distributive Property (continued)

Here are the functions written in algebraic form:

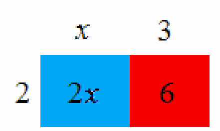
$f(x) = 2(x + 3)$
$g(x) = 2x + 6$

Using the distributive law, we have:

$$2(x+3) = (2 \times x) + (2 \times 3) = 2x + 6$$

so $f(x) = g(x)$, and the two Logo functions are equal.

When you're working on these distributive property problems, it helps to make a picture. For this problem, we could draw a picture like this:



If we look at the red and blue areas as two pieces of a big rectangle, we can see that the area of the whole rectangle must be $2(x + 3)$. But we can also calculate the areas of the blue and red rectangles separately. The blue rectangle is a $2 \times x$, so its area is $2x$. The red rectangle is a 2×3, so its area is 6. Since the area of the big rectangle is the same as the sum of the areas of the red and blue rectangles, we have:

$$2(x+3) = 2x + 6$$

Other Properties

Here are other number properties you can use to help tell if two functions are equal:

Additive identity

$0 + x = x$

Multiplicative identity

$1 \times x = x$

Multiplicative property of zero

$0 \times x = 0$

Substitution property

If 2(a + b) = 3c and a + b = 9, then 2(9) = 3c

Additive property of equality

If $2 = x$ then $2 + 5 = x + 5$

Multiplicative property of equality

If $x = 2$ then $4x = 2(4)$, or $4x = 8$

Operating on Variables

In elementary school, you learned that multiplication is repeated addition. So, for example:

$3 \times 5 = 5 + 5 + 5$

In the same way, we can rewrite the product of a variable and a number as a sum:

$2x = x + x$

More often, however, we find it convenient to rewrite a sum of variables as a product of a number and a variable:

$x + x = 2x$

Another way to think of this is, "If I have an x and I add another x, then all together I have 2 x's".

Using this kind of thinking, it is easy to see that:

$2x + 3x = 5x$ (if I have two x's and then add three more x's, I'll have five x's)

Another way to demonstrate this equality is as follows. Since $2x = x + x$ and $3x = x + x + x$, we could say:

$2x + 3x = (x + x) + (x + x + x) = x + x + x + x + x = 5x$

A third way to demonstrate the equality is to use the distributive property:

$2x + 3x = (2 + 3)x = 5x$

We can also distribute with subtraction:

$11y - 4y = 7y$

Solved Problems

1. *Problem:* Simplify the following expressions with paper and pencil:

   a. $2(x - 4)$
   b. $(x + 2)(x + 3)$
   c. $x + 3(5 + x) - 2x + 1$

   Solution:

   a. Here we apply the distributive property:

   $$2(x-4) = (2 \times x) - (2 \times 4) = 2x - 8$$

   Remember, subtraction isn't commutative or associative, but you can distribute multiplication over subtraction.

   b. For this problem, we need to use the distributive property four times. First, we distribute the $(x + 2)$:

   $$(x+2)(x+3) = [(x+2) \times x] + [(x+2) \times 3]$$

   Next, we use the distributive property two more times to distribute the x over the first $(x + 2)$ and the 3 over the next $(x + 2)$:

   $$[(x+2) \times x] + [(x+2) \times 3] = [(x \times x) + (2 \times x)] + [(x \times 3) + (2 \times 3)]$$

   Simplifying further, we get

   $$[(x \times x) + (2 \times x)] + [(x \times 3) + (2 \times 3)] = x^2 + 2x + 3x + 6$$

   Finally, we use the distributive property to add $2x$ and $3x$:

   $$x^2 + 2x + 3x + 6 = x^2 + 5x + 6$$

   In algebraic notation, the symbols [and] are used group parts of an expression that already uses (and). The symbols [and] are called *brackets*. The symbols (and) are called *parentheses*. In Logo, brackets are used for a different kind of grouping, so expressions like the one above are written with extra parentheses.

Solved Problems (continued)

For example, in Logo, the expression

$$5 \times [(x \times 3) + (2 \times 3)]$$

Would be written as:

5 * ((:x * 3) + (2 * 3))

Lots of brackets and parentheses can be confusing, which is probably why mathematicians came up with abbreviations for multiplication like

3x for $3 \times x$

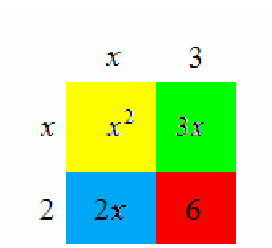
If we rewrite the solution using this abbreviation, we can see how it makes things easier to follow:

$$(x+2)(x+3) = (x+2)x + (x+2)3 = x^2 + 2x + 3x + 6 = x^2 + 5x + 6$$

Using this kind of abbreviation can be very helpful as long as you remember what the abbreviations mean and can convert them to their Logo equivalents.

You might have noticed that we used the associative law to tell us that it was OK to add the $2x$ and the $3x$ together before adding x^2.

We can also think about this problem in terms of rectangles like we did in the section above on the distributive property:



The picture shows that $(x+2)(x+3) = x^2 + 2x + 3x + 6$. In this case, the only work left to do is to add together the $2x$ and the $3x$ to get the final answer: $x^2 + 2x + 3x + 6 = x^2 + 5x + 6$.

Solved Problems (continued)

c. In simplifying expressions, it is often helpful to use the distributive property (if possible) before doing anything else. That's what we'll do here:

$$x + 3(5 + x) - 2x + 1 = x + 3(5) + 3x - 2x + 1 = x + 15 + 3x - 2x + 1$$

The expression 3(5) is another way of writing 3×5. Now we can use the commutative property to move things around:

$$x + 15 + 3x - 2x + 1 = x + 3x - 2x + 15 + 1$$

But wait a minute. Didn't we say that subtraction isn't commutative? What's going on here?

If you look closely, you can see that we actually only moved the 15. Since we can add things on anywhere (addition is commutative), it's OK to add the 15 someplace else.

But why move things around at all? Well, if you remember, we can use the distributive property to add and subtract x's, and we can also add 15 to 1, so moving things around allows us to simplify like this:

$$x + 3x - 2x + 15 + 1 = 4x - 2x + 16 = 2x + 16$$

So the simplified version of $x + 3(5 + x) - 2x + 1$ is $2x + 16$.

Solved Problems (continued)

2. *Problem:* Write the functions given in problem 1 as Logo functions. Next, write your solutions to problem 1 as Logo functions. Compare the problem and solution functions with three different values to make sure they give the same answer.

   Solution:

   For each problem, we can let *f* be the given (problem) function and let *g* be the solution function.

   a. $f(x) = 2(x - 4)$, $g(x) = 2x - 8$

```
to f :x
op 2 * (:x - 4)
end

to g :x
op 2 * :x - 8
end
```

   Here are some tests of the functions:

```
show f 4
0
show g 4
0
show f 8
8
show g 8
8
show f 11
14
show g 11
14
```

Solved Problems (continued)

b. $f(x) = (x + 2)(x + 3)$, $g(x) = x^2 + 5x + 6$

```
to f :x
op (:x + 2) * (:x + 3)
end

to g :x
op :x * :x + 5 * :x + 6
end
```

There is no Logo primitive that corresponds to x^2. We could write our own:

```
to square :x
op :x * :x
end
```

Then we could rewrite operation `g` as

```
to g :x
op (square :x) + 5 * :x + 6
end
```

We need the parentheses around `square :x`. Otherwise, Logo will think that the input to `square` is supposed to be `:x + 5 * :x + 6` instead of just `:x`. If this isn't clear to you, you can try running `g` with and without the parentheses while you have Trace turned on.

```
show f 0
6
show g 0
6
show f 235
56406
show g 235
56406
show f 2
20
show g 2
20
```

Solved Problems (continued)

c. $f(x) = x + 3(5 + x) - 2x + 1$, $g(x) = 2x + 16$

```
to f :x
op :x + 3 * (5 + :x) - 2 * :x + 1
end

to g :x
op 2 * :x + 16
end
```

```
show f 0
16
show g 0
16
show f 2
20
show g 2
20
show f 10
36
show g 10
36
```

Since the functions are equal, these very different-looking programs will always give you the same output when they are each given the same input. Pretty cool, huh?

Supplementary Problems

1. *Problem:* Simplify the following expressions with paper and pencil:

a. $(9x + 6) + (x - 4)$
b. $(x + 4)(x + 1)$
c. $2 + 7(3 + x) + 1$

Solution:

a. $(9x + 6) + (x - 4)$

$= 9x + 6 + x - 4$	*associative property*
$= 9x + x + 6 - 4$	*commutative property*
$= 10x + 2$	*distributive property*

b. $(x + 4)(x + 1)$

$= (x + 4)x + (x + 4)1$	*distributive property*
$= (x + 4)x + (x + 4)$	*multiplicative identity*
$= x^2 + 4x + x + 4$	*distributive property*
$= x^2 + 5x + 4$	*distributive property*

c. $2 + 7(3 + x) + 1$

$= 2 + 7(3) + 7x + 1$	*distributive property*
$= 2 + 21 + 7x + 1$	*7(3) = 21*
$= 7x + 2 + 21 + 1$	*commutative property*
$= 7x + 23 + 1$	*2 + 21 = 23*
$= 7x + 24$	*23 + 1 = 24*

Supplementary Problems (continued)

2. *Problem:* Write the functions given in problem 1 as Logo functions. Next, write your solutions to problem 1 as Logo functions. Compare the problem and solution functions with three different values to make sure they give the same answer.

   Solution:

   For each of these solutions, you can run tests of the functions with any three inputs you choose. We can define a function `square` that multiplies a number by itself:

```
to square :x
op :x * :x
end
```

   For each problem in number 1, we can let f be the given (problem) function and let g be the solution function.

   a. $f(x) = (9x + 6) + (x - 4)$, $g(x) = 10x + 2$

```
to f :x
op (9 * :x + 6) + (:x - 4)
end

to g :x
op 10 * :x + 2
end
```

   b. $f(x) = (x + 4)(x + 1)$, $g(x) = x^2 + 5x + 4$

```
to f :x
op (:x + 4) * (:x + 1)
end

to g :x
op (square :x) + 5 * :x + 4
end
```

Supplementary Problems (continued)

c. $f(x) = 2 + 7(3 + x) + 1$, $g(x) = 7x + 24$

```
to f :x
op 2 + 7 * (3 + :x) + 1
end

to g :x
op 7 * :x + 24
end
```

The Author

TJ Leone owns and operates Leone Learning Systems, Inc., a private corporation that offers tutoring and educational software. He has a BA in Math and an MS in Computer Science, both from the City College of New York. He spent two years in graduate studies in education and computer science at Northwestern University, and six years developing educational software there. He is a former Montessori teacher and currently teaches gifted children on a part time basis at the Center for Talent Development at Northwestern University in addition to his tutoring and software development work. His web site is http://www.leonelearningsystems.com.