1. Algebraic solution of Project 1.12

Write a function that expresses the idea of repeated multiplication with a generic decrement and length. In other words, write a function rep_mult **that takes three inputs** n**,** d**, and** l**. The output of** $rep\_mult(n, d, l)$ **should be**

n(n-d)(n-2d)(n-3d)...(n-(l-1)d)

so that $rep\_mult(15, 2, 6)$ **will output**

$15(15-2)(15-2\cdot 2)(15-3\cdot 2)(15-4\cdot 2)(15-5\cdot 2) = 15\cdot 13\cdot 11\cdot 9\cdot 7\cdot 5$**, or 675675.**

The function we write must involve repeated multiplication. What functions have we seen that involve repeated multiplication? In the problem statement, reference is made to the factorial and power functions (page 32).It's hard to see how to make a direct path from those functions to *rep_mult*. What other related functions do we know? How about *prod*? *Prod* involves multiplying together a given number of factors, and so does *rep_mult*. Can we relate *rep_mult* and *prod*? It's usually easier to start with concrete examples, so let's first consider the concrete example given in Project 1.12:

$$15(15-2)(15-2\cdot 2)(15-3\cdot 2)(15-4\cdot 2)(15-5\cdot 2)$$

The $(15-2\cdot 2)$, $(15-3\cdot 2)$, $(15-4\cdot 2)$, and $(15-5\cdot 2)$ look very similar. We could write those four factors as:

$$\prod_{k=2}^{5} 15-2k$$

But what about the first two factors, 15 and $(15-2)$. Let's see. $(15-2)$ looks more like the other factors, so let's tackle that one first. Is there a pattern in the other terms that can help us? Starting from the end, we have $(15-5\cdot 2)$, $(15-4\cdot 2)$, $(15-3\cdot 2)$, and $(15-2\cdot 2)$. The next logical factor in this pattern should be $(15-1\cdot 2)$. But wait a minute: $(15-1\cdot 2) = (15-2)$. So we can rewrite our original equation for $rep\_mult(15, 2, 6)$ as

$$15(15-1\cdot 2)(15-2\cdot 2)(15-3\cdot 2)(15-4\cdot 2)(15-5\cdot 2)$$

Now we can back up our pattern one more step to get $(15-0\cdot 2) = 15$. Now we have

$$(15-0\cdot 2)(15-1\cdot 2)(15-2\cdot 2)(15-3\cdot 2)(15-4\cdot 2)(15-5\cdot 2)$$

and our *prod* becomes

$$\prod_{k=0}^{5} 15-2k$$

Suppose we want a function that lets us choose the number of factors we want in our $\prod$. Let's use the variable l for the number of factors and write f as a function of l:

$$f(l) = \prod_{k=0}^{l-1} 15 - 2k$$

Notice that our lower bound for k is 0 and upper bound is $l - 1$. So, for example, when $l = 6$, k takes on the values of 0, 1, 2, 3, 4, and 5, to give us the six factors

$$(15 - 0 \cdot 2)(15 - 1 \cdot 2)(15 - 2 \cdot 2)(15 - 3 \cdot 2)(15 - 4 \cdot 2)(15 - 5 \cdot 2)$$

We could make some further progress toward a version of *rep_mult* using this approach, but there's another way of looking at functions that is very powerful in both computer science and mathematics. We will now turn to that approach.

Consider another function g. We'll define g as follows:

(1) $\quad g(0) = 1$

For any integer $l > 0$,

(2) $\quad g(l) = (15 - (l - 1) \cdot 2) \cdot g(l - 1)$

This is a *recursively defined function*. The best way to decode such functions is to plug in some values and see what results you get.

For example, we know that $g(0) = 1$, but what is $g(1)$? The first thing to consider is that $1 > 0$, so we need to use equation (2). Let's try plugging in 1 for l in that equation:

$$\begin{aligned} g(1) &= (15 - (1 - 1) \cdot 2) \cdot g(1 - 1) \\ &= (15 - 0 \cdot 2) \cdot g(0) \end{aligned}$$

Now what do we do? Well, we know that $g(0) = 1$, so we can replace $g(0)$ with 1:

$$\begin{aligned} g(1) &= (15 - 0 \cdot 2) \cdot g(0) \\ &= (15 - 0 \cdot 2) \cdot 1 \\ &= (15 - 0 \cdot 2) \end{aligned}$$

Now that we have a value for $g(1)$, we can calculate $g(2)$:

$$\begin{aligned} g(2) &= (15 - (2 - 1) \cdot 2) \cdot g(2 - 1) \\ &= (15 - 1 \cdot 2) \cdot g(1) \\ &= (15 - 1 \cdot 2) \cdot (15 - 0 \cdot 2) \end{aligned}$$

OK, so far we have $g(0) = 1$, $g(1) = (15-0\cdot 2)$, and $g(2) = (15-1\cdot 2)\cdot(15-0\cdot 2)$. What about $g(3)$?

$$\begin{aligned} g(3) &= (15-(3-1)\cdot 2)\cdot g(3-1) \\ &= (15-2\cdot 2)\cdot g(2) \\ &= (15-2\cdot 2)\cdot(15-1\cdot 2)\cdot(15-0\cdot 2) \end{aligned}$$

There's another way to expand out the factors in the function g. Let's look back at the definition:

(3) $\quad g(0) = 1$

For any integer $l > 0$,

(4) $\quad g(l) = (15-(l-1)\cdot 2)\cdot g(l-1)$

Let's try to plug in $l = 3$ directly:

$$\begin{aligned} g(3) &= (15-(3-1)\cdot 2)\cdot g(3-1) \\ &= (15-2\cdot 2)\cdot g(2) \end{aligned}$$

Next, we replace $g(2)$ using the definition of $g(l)$ in (4):

$$\begin{aligned} g(3) &= (15-2\cdot 2)\cdot g(2) \\ &= (15-2\cdot 2)\cdot(15-(2-1)\cdot 2)\cdot g(2-1) \\ &= (15-2\cdot 2)\cdot(15-1\cdot 2)\cdot g(1) \end{aligned}$$

In the same way, we replace $g(1)$ with its definition:

$$\begin{aligned} g(3) &= (15-2\cdot 2)\cdot(15-1\cdot 2)\cdot g(1) \\ &= (15-2\cdot 2)\cdot(15-1\cdot 2)\cdot(15-(1-1)\cdot 2)\cdot g(1-1) \\ &= (15-2\cdot 2)\cdot(15-1\cdot 2)\cdot(15-0\cdot 2)\cdot g(0) \end{aligned}$$

From equation (3), we know that $g(0) = 1$, so we can replace $g(0)$ with 1:

$$\begin{aligned} g(3) &= (15-2\cdot 2)\cdot(15-1\cdot 2)\cdot(15-0\cdot 2)\cdot g(0) \\ &= (15-2\cdot 2)\cdot(15-1\cdot 2)\cdot(15-0\cdot 2)\cdot 1 \\ &= (15-2\cdot 2)\cdot(15-1\cdot 2)\cdot(15-0\cdot 2) \end{aligned}$$

You should recognize by now that $g(l) = f(l)$ for all integers > 0, and is almost what we need for *rep_mult*. For *rep_mult* we need to add the ability to choose different values besides 15 or 2, so we need to add a couple of variables:

$$rep\_mult(n, d, 0) = 1 \quad (5)$$

For any integer $l > 0$,

$$rep\_mult(n, d, l) = (n - (l - 1) \cdot d) \cdot rep\_mult(n, d, l - 1) \quad (6)$$