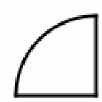
An Alternative Derivation of Lateral Area of a Cone

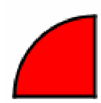
Below is a rough sketch of a presentation of an alternative derivation of the lateral area of a cone. I worked this out after the lecture on Surface Area of Solids because I couldn't get the triangle explanation. I tried to give an idea of how I might show this derivation to kids, but I haven't tried it out so I don't know if it would work. If you ever have any thoughts on whether or how this could become something useful for an elementary classroom, I'd love to hear from you.

Overview of the Derivation

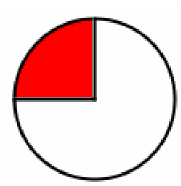
We start by rolling the cone like Allyn did to make a sector:



What's different in this derivation we think of the sector as a fraction of a circle instead of thinking of it as a triangle. First, let's color in the sector to keep track of it:



Now let's complete the circle:

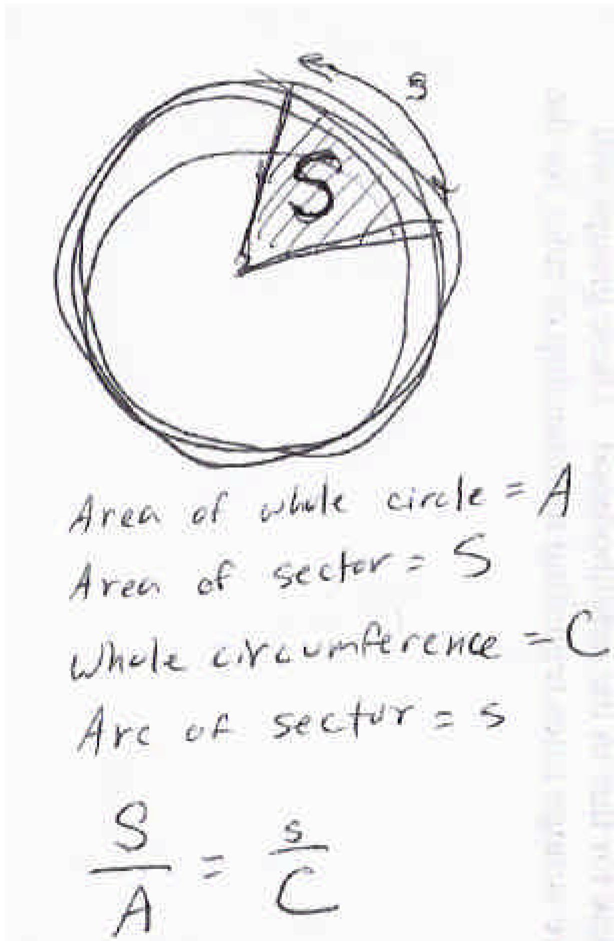


We know the radius of this circle is the same as the slant height, so we can figure out the area of the circle. Suppose we know that the sector is one fourth of the circle. Then we just need to multiply the circle area by one fourth, and we have the area of the sector.

The trick, then, is to figure out what fraction of the circle is represented by the sector. This requires that the children understand a couple of things about cones and circles, so we need a couple of setup lessons before we get to the derivation.

Setup 1

Purpose: To establish that the following holds for all sectors and circles:

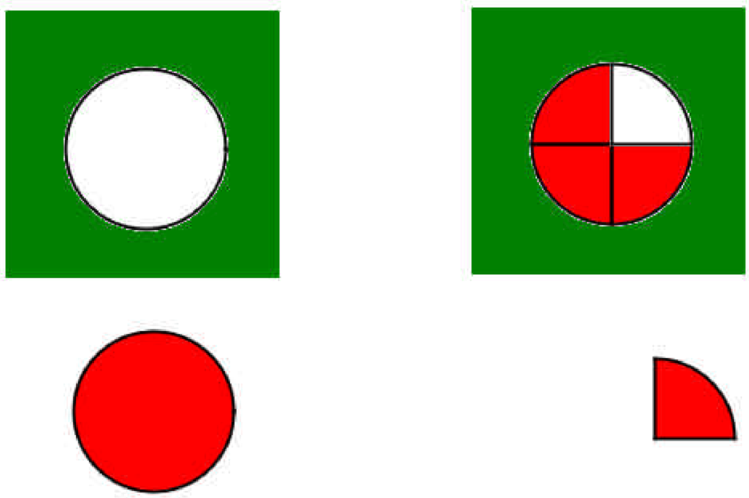


In English, this means that if the sector's arc is three sevenths of the circumference, then the sector is three sevenths of the surface of the circle. This is true for any proper fraction (i.e., if the sector's arc is half the circumference, then the sector covers half the surface of the circle).

Kids have lots of opportunities to be exposed to this idea in work with my Circular Reasoning software, but below is a presentation that could be done with fraction circles to get the idea across.

Presentation:

Take out a whole fraction circle and a quarter circle.



[Point to whole circle]

Here we have a whole circle…

[Point to quarter circle]

…and here we have a sector.

What is the relationship between the area of the sector and the area of the whole circle? Right, the area of the sector is one fourth the area of the circle.

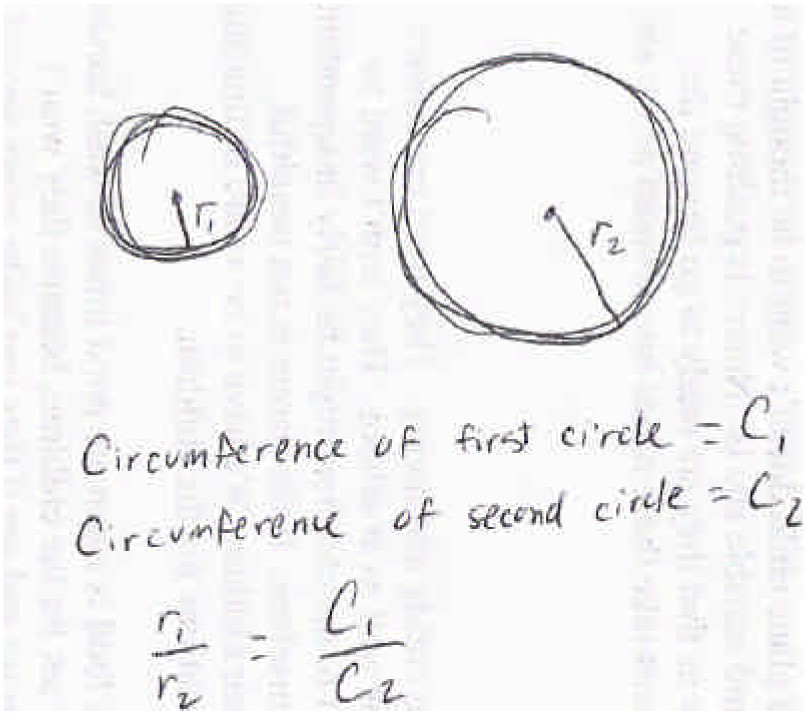
Now let's look at just the arc of the sector and the circumference of the circle.

[You want to get across the idea that the arc of the sector is one fourth of the circumference of the circle. You might do this by pointing out the four equal arcs defined by the four fourths, or you might roll the whole circle on a line like we did in the Area of a Circle activity and then show how you can roll the quarter arc four times to cover the length covered by the whole circle.]

[Next you might ask them about other sectors and corresponding arcs (one third, one eighth and so on) and then ask about sectors that make up fractions with numerators greater than one, so they see that if the sector is 4/7 of a circle then the arc defined by that sector is 4/7 of the circumference. They should also see that this works the other way around, i.e., if the arc is 2/5 of the circumference then the sector defined by that arc has 2/5 the area of the circle.]

Setup 2

Purpose: To show that the following is true for any two circles:



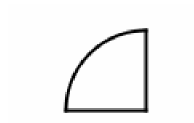
[You could do this a couple of ways. Sensorially, you might use the geometry sticks to draw one circle with a radius that twice as big as the other. Then you could lay string around the circumference of each circle and measure the string to get the circumference, and then compare the circumferences. You could also confirm the relationship algebraically like this:

$$\frac{C_1}{C_2} = \frac{2\pi r_1}{2\pi r_2} = \frac{r_1}{r_2}$$

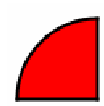
with whatever intermediate steps the child needs to understand the equations.]

Derivation

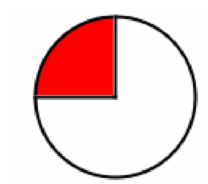
OK, now we're ready to do the derivation. We start like Allyn did, rolling the cone to define a sector:



Looks like we started to make a circle. What if we did go all the way around and make a circle? Let's color in our sector first so we remember that's the area we want to find.



Now let's complete the circle.



What part of the cone would be radius of that circle? Right, the slant height of the cone. So what is the area of the whole circle? Right, two pi times the slant height squared.

$p h_s^2$

I put a little s down here by the h so we remember that we are talking about the slant height.

But we don't want the area of the whole circle. We just want the area of the sector colored in red. What does that sector remind you of? It looks kind of like a fraction piece from the fraction circles, doesn't it?

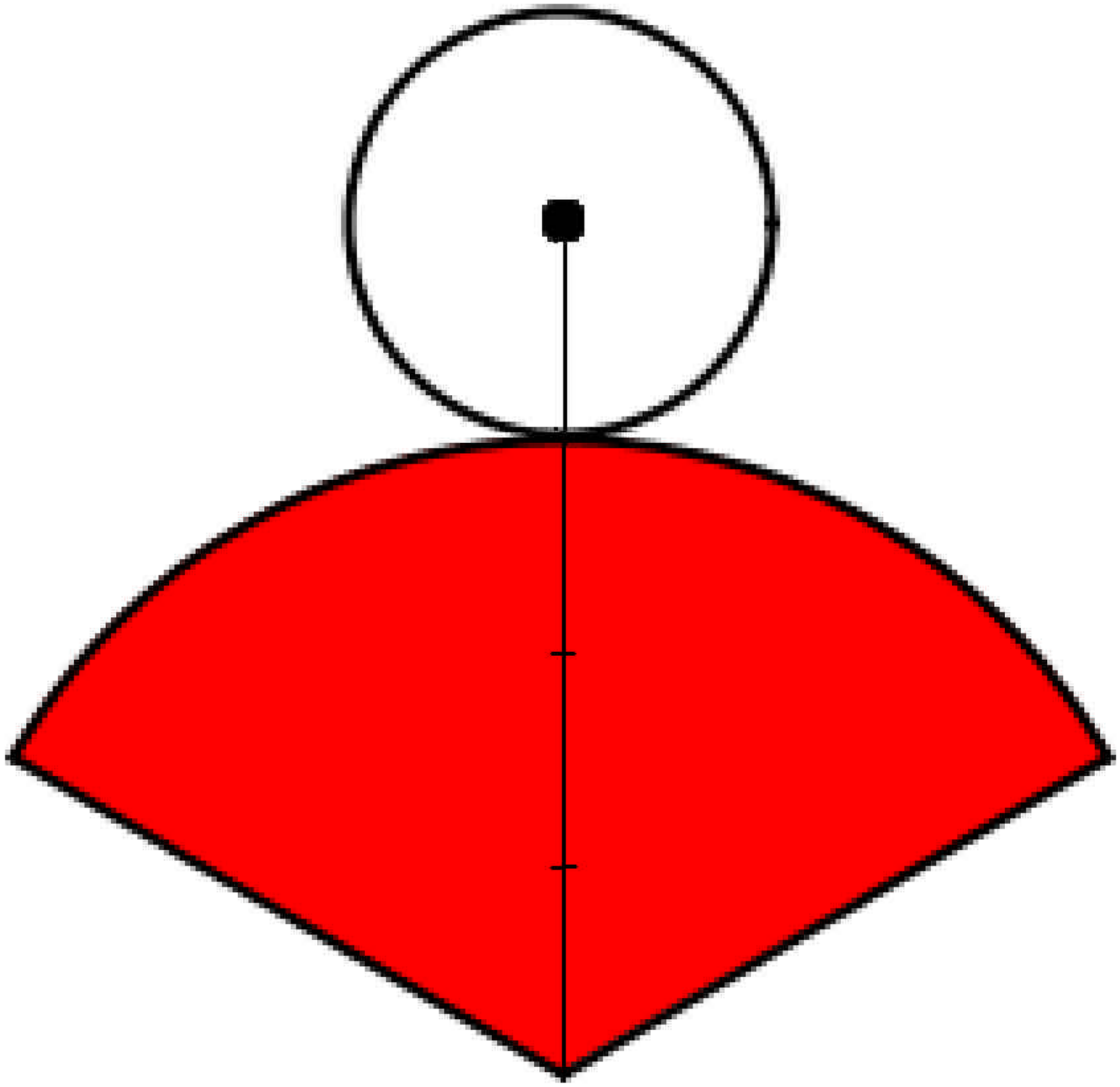
What if we knew for sure that the sector was one fourth of the circle? We know that the area of the circle is $p h_s^2$. *How would we find the area of the sector? Right, we could just multiply one fourth times* $p h_s^2$ *to take one fourth of the whole circle.*

So if we can just figure out what fraction we have here, we can figure out the area of the sector, which is the same as the lateral area of our cone. Let's write it like this:

$$L.A. = p h_s^2 \times F$$

I'm using F to stand for the fraction we need to find.

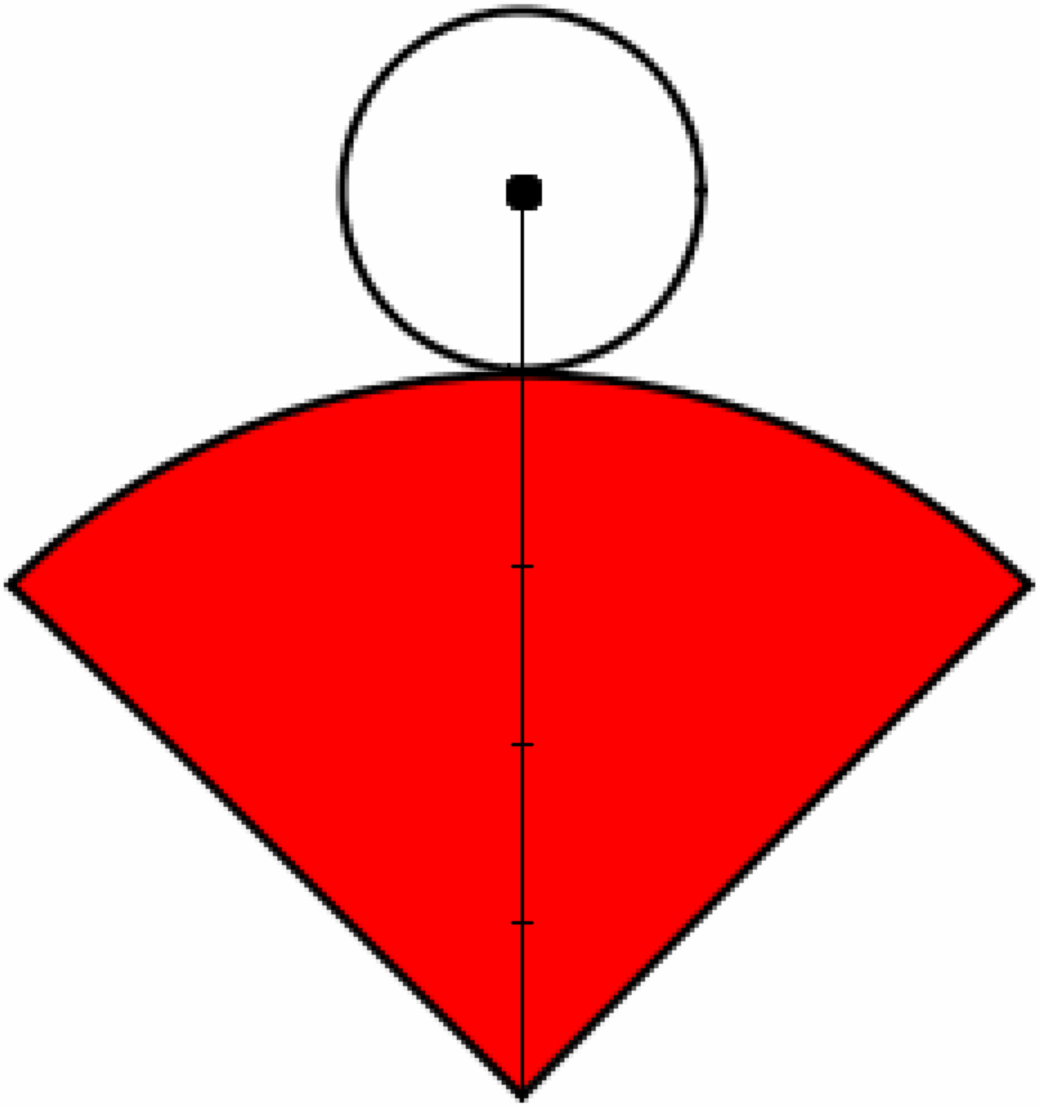
First, let me show you some cut-outs you can use to make some different cones.



I'm using the radius of the base as a unit of measure to measure the slant height. Here I've drawn the radius of the base, and how many radii of the base equal the radius of the sector? Three. And the fraction made by the red sector looks like about how much? About one third, doesn't it? Let's confirm that with the one third secto from the fraction pieces.

[Superimpose the one third piece from the fraction circles on the paper sector to show they are both a third of a circle. Let children assemble the paper cone.]

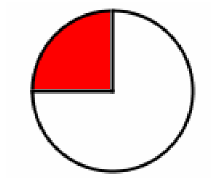
How many radii make up the slant height of this cone? Four. So what fraction do you think that red sector is? One fourth? Let's see.



[Superimpose one fourth from the fraction circles onto the red sector to show that the red sector is one fourth. Let the children put the cone together]

This cone looks like it's the same shape as the cone from our solid geometry materials, doesn't it? When we rolled out the sector for that cone, what fraction of a circle did it look like?

[Bring out the sector drawn by rolling the cone]



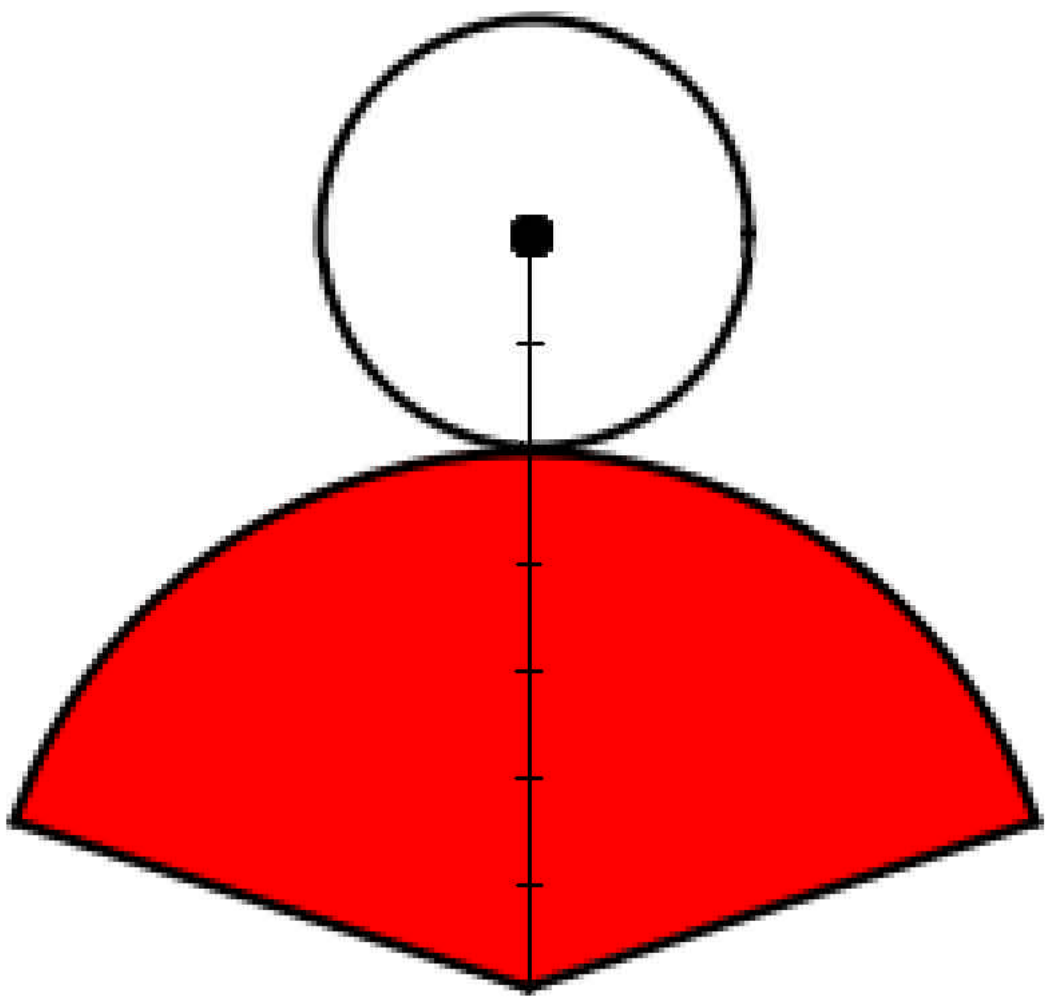
This looks like it might also be one fourth. Let's verify that.

[Use fraction circle to verify]

How many radii of the base do you think it would take to measure the slant height? Let's see.

[For the solid cone that was used in our class demonstration, the radius of the base is 5cm and the slant height is 20cm, so we can show that four base radii are equivalent to the slant height]

This one is a little different, because I have a different unit of measure. Two of the units measure the radius of the base, and five of those same units measure the slant height. What fraction do you think that sector is?



[Help children see that the sector is two fifths, verifying with fraction circles.]

So it looks like the radius of base divided by the slant height gives us the fraction of the sector. I wonder why that is?

Let's see. We know that the length of this arc…

[Indicate the arc of the sector]

Is as long as what? Right, it's as long as the circumference of the base. So if we divide the circumference of the base by the circumference of this big circle we've made, we'll know what fraction of the circumference is made by this arc. Will that help us?

Well, if we find out what fraction of the circumference is made by this arc, we'll know what fraction of the circle's surface is covered by this sector, and that's what we need to know, right? That the F in our equation that we need to figure out.

OK, so the whole circle would have a circumference of two times pi times the slant height. I'll use a little s to big circle parts and a little b for base parts.

$$C_s = 2\pi h_s$$

We also said that the arc we drew has a length of the circumference of the base

$$C_b = 2\pi r_b$$

What fraction tells us how much of the circle is drawn by our arc?

$$F = \frac{C_b}{C_s} = \frac{2\pi r_b}{2\pi h_s} = \frac{r_b}{h_s}$$

Remember that this fraction also tells us what part of our big circle is covered by our sector. So now we have our F, and we can write this:

$$L.A. = \pi h_s^2 \times \frac{r_b}{h_s} = \pi h_s r_b$$

Since $C_b = 2\pi r_b$*, we can also say that*

$$L.A. = \pi h_s r_b = \pi r_b h_s = \frac{2\pi r_b}{2} h_s = \frac{C_b}{2} h_s$$