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Binomial Square Explained

A teacher's guide

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Montessori's binomial cube was designed to provide an intuitive introduction to the cubing of a binomial. To prepare the child to move into abstraction, teachers should have a deep understanding of the pattern that appears on four faces of the binomial cube—the binomial square.

This guide was written to show teachers how the binomial square can be used to represent the formula

$$(p + q)^2 = p^2 + 2pq + q^2$$

The Binomial Square

On the cover of the box for the binomial cube there is a painted pattern that looks like (Figure 1).
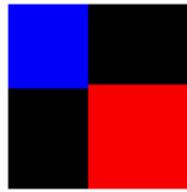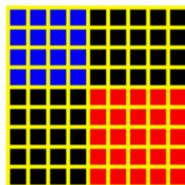

• Figure 1.

This same pattern appears on four faces of the constructed cube.

Suppose we know that the blue square is 4 units long on each side, and the red square is 5 units long on each side. In the figure below (Figure 2), a grid has been superimposed on the binomial square to show these units.

To find the area of the binomial square, we could simply count the number of squares in the grid (there are eighty-one). There are also two important ways to calculate this area using the basic formula:

• Figure 2.

Area = width x height

1. We could start by calculating the length of the sides of the binomial. We can see from the figure that the total width of the binomial square must be $4 + 5 = 9$. The same is true for the height. Since we know that the width and height of the binomial square are both 9, we can calculate the area as $9 \times 9 = 81$. Notice that we can also write 9×9 as 9^2 or $(4 + 5)^2$.

2. We could also calculate individually the areas of the blue and red squares and the two black rectangles and then add these areas together:

Blue square	$4^2 = 16$
2 Black rectangles	$2 \times 4 \times 5 = 40$
Red square	$5^2 = 25$

$16 + 40 + 25 = 81$. Another way to express this is:
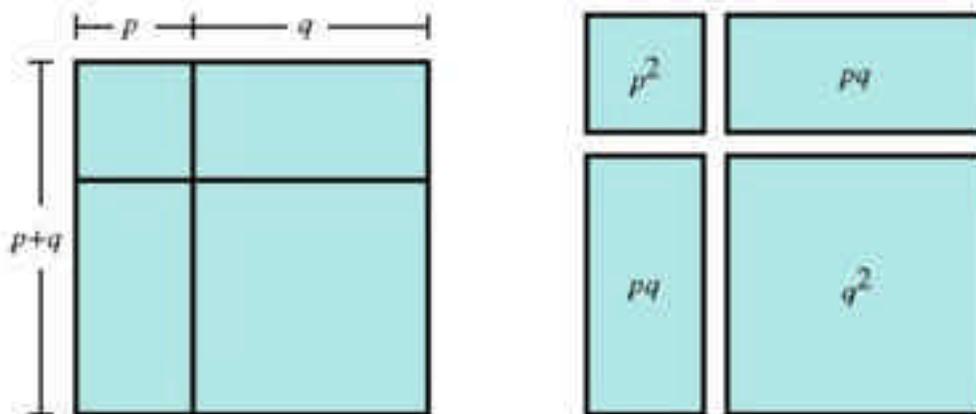
$4^2 + 2 \times 4 \times 5 + 5^2 = 81$.

Note that each black rectangle has an area of $4 \times 5 = 20$. We have to multiply this product by 2 to get the area of the 2 rectangles. That's how we got $2 \times 4 \times 5$.

We can show the equivalence of the two methods of calculating the area like this:

$$(4 + 5)^2 = 4^2 + 2 \times 4 \times 5 + 5^2$$

The Algebraic Formula

. Using algebra, we can write the square of the binomial in a general way (Figure 3):

- Figure 3.

$$(p + q)^2 = p^2 + 2pq + q^2$$

The traditional algebraic derivation of this formula is:

$$(p + q)^2 = (p + q)(p + q)$$ Definition of squaring: $x^2 = x$ times x
$$= p(p + q) + q(p + q)$$ Distributive property: (a + b)c = ac + bc
$$= p^2 + pq + qp + q^2$$ Distributive property: a(b + c) = ab + ac
$$= p^2 + pq + pq + q^2$$ Commutative property of multiplication: qp = pq
$$= p^2 + 2pq + q^2$$ Definition of multiplication as repeated addition

Higher powers of the binomial can be evaluated in similar ways. The binomial cube is a geometric expression of the 3rd power of a binomial $(p + q)^3$. There are also Montessori materials for the 4th and 5th powers of a binomial. The trinomial cube is a geometric expression of $(p + q + r)^3$.

See *The Geometry of the Binomial Theorem* (Banchoff, 2000) for more details.

References

Banchoff, T. (2000). *The Geometry of the Binomial Theorem*. Retrieved August 8, 2006, from
http://www.mathaware.org/mam/00/master/essays/B3D/2/binomial.html

The Author

TJ Leone owns and operates Leone Learning Systems, Inc., a private corporation that offers tutoring and educational software. He has a BA in Math and an MS in Computer Science, both from the City College of New York. He spent two years in graduate studies in education and computer science at Northwestern University, and six years developing educational software there. He is a former Montessori teacher and currently teaches gifted children on a part time basis at the Center for Talent Development at Northwestern University in addition to his tutoring and software development work. His web site is http://www.leonelearningsystems.com.